RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2617

Statistics 5

Tuesday **28 MAY 2002** Afternoon 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The approximate allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 60.

This question paper consists of 3 printed pages and 1 blank page.

[Turn over

1 **(i)** The random variable X has the binomial distribution with parameters n and p. Derive the probability generating function of X and hence obtain the mean and variance of X. [9]

(ii) The editor of an academic journal is concerned about an apparent increase in the proportion of articles submitted for publication that are rejected as being unsatisfactory. For many years this proportion has been steady at around one quarter of all articles submitted. However, in the last year, out of a total of 68 submitted articles, 23 were rejected as unsatisfactory. The articles that were submitted to the journal may be regarded as a random sample from a population of potential articles. Test at the 10% level of significance whether the population proportion rejected as unsatisfactory now exceeds 0.25, stating clearly your null and alternative hypotheses and your conclusion. [11]

2 The discrete random variable X takes the values –1, 0 and 1, each with probability $\frac{1}{3}$.

(i) Write down the values of μ, the mean, and σ^2, the variance, of X. [2]

(ii) Find the moment generating function of X. [2]

(iii) Let Z denote the standardised mean for a random sample of n independent observations on X, i.e.

$$Z = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

where $\overline{X}$ is the sample mean. Stating carefully any general results about moment generating functions that you use, show that the moment generating function of Z is given by $\mathrm{M}(\theta)$ where

$$\mathrm{M}(\theta) = \left\{\frac{1}{3}\left(1 + \mathrm{e}^{\frac{\theta\sqrt{3}}{\sqrt{2n}}} + \mathrm{e}^{-\frac{\theta\sqrt{3}}{\sqrt{2n}}}\right)\right\}^n$$ [7]

(iv) By expanding the exponential functions in $\mathrm{M}(\theta)$, show that, for large n,

$$\mathrm{M}(\theta) \approx \left(1 + \frac{\frac{1}{2}\theta^2}{n}\right)^n.$$ [5]

(v) Use the result $\mathrm{e}^y = \lim_{n\to\infty}\left(1 + \frac{y}{n}\right)^n$ to show that $\mathrm{M}(\theta)$ tends to $\mathrm{e}^{\theta^2/2}$ as n tends to ∞.

Deduce the approximate distribution of Z for large n. [4]

3 Financial analysts are studying the variability in the performances of a large number of apparently similar companies. The values of a particular financial indicator for a sample of 11 companies in one country are as follows.

18.6 16.4 12.2 21.3 18.8 19.4 17.7 20.6 22.8 14.9 23.1

The values of this indicator for a sample of 13 companies in another country are as follows.

22.6 22.1 21.4 22.8 19.2 22.5 22.7 20.6 24.8 21.3 19.6 20.9 22.2

(i) Examine at the 5% level of significance whether the underlying variances in the two countries may be assumed equal. [8]

(ii) Provide a one-sided 95% confidence interval giving an upper bound for the underlying variance in the first country. [5]

(iii) State carefully the distributional results underlying your analyses in parts **(i)** and **(ii)**. State carefully the assumptions required for these distributional results to apply. [7]

4 A company makes heavy-duty waterproof clothing. Part of the manufacturing process consists of spraying a polymer on to a synthetic fibre. The water-absorbent quality of the fibre after this spraying is routinely measured during the manufacturing process. Low values of this measure are desirable.

In the existing process, it is found that the behaviour of the measure is well modelled by the Normal distribution with mean 48.6 and standard deviation 2.4.

An experimental process is being developed. It has been established that the corresponding model for this process is again Normal and with the same standard deviation, but its mean μ is as yet unknown. It is required to examine the null hypothesis H_0: $\mu = 48.6$ against the alternative hypothesis H_1: $\mu < 48.6$, using the customary significance test based on the mean $\overline{X}$ of a random sample of size n. To avoid unnecessary costs of changing from the existing process, it is required that the probability of rejecting H_0 if in fact μ is 48.6 should be at most 3%. If on the other hand μ is in fact 45.0, it is required that the probability of accepting H_0 should be at most 2%.

(i) Find an expression for the critical value of $\overline{X}$ and show that the least sample size that will meet the requirements is 7. [14]

(ii) Taking $n = 7$, derive an expression for the power function of the test in the form

$$P(Z < a - b\mu)$$

where $Z \sim N(0, 1)$ and a and b are constants to be determined. Hence verify that the requirement when $\mu = 45.0$ is met. [6]

Mark Scheme

Qu 1

(i) $X \sim B(n, p)$

Pgf $= E[t^X]$ (M1)

$= \sum_{x=0}^{n} t^x \binom{n}{x} p^x q^{n-x} = \sum \binom{n}{x} (pt)^x q^{n-x}$ (1)

$= (q + pt)^n$ (1) Beware of unexplained over-ambitious write-down

$\mu = G'(1)$ (M1)

$G'(t) = np(q+pt)^{n-1}$ (1) $\therefore \mu = np$ (1)

$\sigma^2 = G''(t) + \mu - \mu^2$ (M1)

$G''(t) = n(n-1)p^2(q+pt)^{n-2}$ (1)

$\therefore \sigma^2 = n(n-1)p^2 + np - n^2p^2 = np - np^2 = npq$ (1) **9**

(ii) $H_0 : p = 0.25$ (1)

$H_1 : p > 0.25$ (1)

where $p = P$(article is rejected as unsatisfactory) (1)

23 out of 68 are rejected. Test statistic is

$$\frac{22\frac{1}{2} - 17}{\sqrt{68 \times 0.25 \times 0.75}} = \frac{5.5}{3.5707} = 1.54(03)$$

or

$$\frac{0.33088 - 0.25}{\sqrt{\dfrac{0.25 \times 0.75}{68}}} = \frac{0.08088}{0.0525} = 1.54(03)$$

Numerator (M2); allow (M1) and f.t. if 23 or 0.3382 used.

Denominator (M1).

Value (A1) (1.68(03) as f.t. from 23.)

Refer to N(0,1). (1) No FT.

Critical point is 1.282. (1) No FT.

Significant. (1)

Seems proportion of unsatisfactory articles has gone up (1) **11**

Qu 2

X	−1	0	1
prob	1/3	1/3	1/3

(i) $\mu = 0$ (1) $\sigma^2 = \frac{2}{3}$ (1) **[2]**

(ii) $M_X(\theta) = E[e^{X\theta}] = e^{-\theta}\cdot\frac{1}{3} + e^{0}\cdot\frac{1}{3} + e^{\theta}\cdot\frac{1}{3}$ (M1)

$= \frac{1}{3}(1 + e^{\theta} + e^{-\theta})$ (1) **[2]**

(iii) "Convolution theorem" to obtain $M_{\Sigma X}(\theta)$:

mgf of sum = product of individual mgfs (if independent) (1)

$\therefore M_{\Sigma X}(\theta) = \left\{\frac{1}{3}(1+e^{\theta}+e^{-\theta})\right\}^n$. (1)

"linear transformation" result to obtain $M_{\bar{X}}(\theta)$ (1)

$M_{\bar{X}}(\theta) = \left\{\frac{1}{3}(1+e^{\theta/n}+e^{-\theta/n})\right\}^n$ (1)

(M1) (Need not be stated explicitly) → linear transformation result again, for $Z = \sqrt{\frac{3n}{2}}(\bar{X}-0)$: (1)

$M_Z(\theta) = \left\{\frac{1}{3}\left(1 + e^{\frac{\theta\sqrt{3}}{\sqrt{2n}}} + e^{-\frac{\theta\sqrt{3}}{\sqrt{2n}}}\right)\right\}^n$ (1) beware printed answer **[7]**

(iv) $M(\theta) = \Big\{\frac{1}{3}\Big(1$

(1) → $+ 1 + \frac{\theta\sqrt{3}}{\sqrt{2n}} + \frac{3\theta^2}{2n.2} +$ terms in $n^{-3/2}, n^{-2}, \ldots$

(1) → $+ 1 - \frac{\theta\sqrt{3}}{\sqrt{2n}} + \frac{3\theta^2}{2n.2} +$ terms in $n^{-3/2}, n^{-2}, \ldots \Big)\Big\}^n$

(CANCEL) (1) NEGLECT THESE FOR LARGE n (1)

$\approx \left\{\frac{1}{3}\left(3 + \frac{3\theta^2}{2n}\right)\right\}^n = \left\{1 + \frac{\frac{1}{2}\theta^2}{n}\right\}^n$ (1) **[5]**

(v) This $\to e^{\theta^2/2}$ (1)

$e^{\theta^2/2}$ is mgf of $N(0,1)$ (1)

Mgfs are unique (even in this limiting process) (1)

$\therefore$ (approx) $Z \sim N(0,1)$ (1) **[4]**

Qu 3

(i) First sample: ($\bar{x} = 18.71$) $s_{n-1}^2 = 3.317^2 = 11.003$ (B1) $n = 11$

($s_n^2 = 3.163^2 = 10.003$)

Second sample: ($\bar{y} = 21.75$) $s_{n-1}^2 = 1.484^2 = 2.201$ (B1) $n = 13$

($s_n^2 = 1.425^2 = 2.032$)

Test statistic $= \dfrac{11.003}{2.201} = 4.999$ [allow 5.0 !] (1) FT candidate's values

Refer to $F_{10,12}$ (1) (1) No FT.

Upper $2\frac{1}{2}$% point is 3.37. (1) [Note for examiner: $F_{12,10}$ point is 3.62] No FT.

Significant. (1)

Suggests underlying variances differ (and higher in first country). (1) **8**

(ii) Use of χ^2_{10} (1) (1)

Lower 5% point is 3.94. (1)

∴ one-sided CI is given by $3.94 < \dfrac{10 \times 11.003}{\sigma^2}$ (1) FT can's value

i.e. $\sigma^2 < 27.93$ (1) **5**

(iii) $\dfrac{S_1^2}{S_2^2} \sim F_{n_1-1,\, n_2-1}$ if $\sigma_1^2 = \sigma_2^2$ (2) may be subdivided

$\dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$ (2) may be subdivided **7**

Underlying Normality (1) in each country (1). Samples are random (1).

Qu 4

(i) Usual test is based on comparing $Z = \dfrac{\bar{X} - 48.6}{2.4/\sqrt{n}}$ with $N(0,1)$. (M1), likely to be implicit in later work

We require

$$0.03 = P(\text{reject } \mu = 48.6 \mid \mu = 48.6) \quad \text{(M1)}$$

$$= P(Z < k \mid Z \sim N(0,1)) \quad \text{(M1)}$$

$$= P(Z < -1.881) \quad \text{(B1)}$$

$\Rightarrow$ reject H_0 if $\dfrac{\bar{x} - 48.6}{2.4/\sqrt{n}} < -1.881$ (M1) accept write-down of this for all marks thus far

ie if $\bar{x} < 48.6 - \dfrac{4.5144}{\sqrt{n}}$ (1) **6**

We require

$$0.02 = P(\text{accept } \mu = 48.6 \mid \mu = 45.0) \quad \text{(M1)}$$

$$= P\left(\bar{X} > 48.6 - \frac{4.5144}{\sqrt{n}} \,\middle|\, \bar{X} \sim N\left(45.0, \frac{2.4^2}{n}\right)\right) \quad \text{(M1), (M1)}$$

$$= P\left(N(0,1) > \frac{3.6 - \frac{4.5144}{\sqrt{n}}}{2.4/\sqrt{n}}\right)$$

might ↑ be implicit

$$= P(N(0,1) > 2.054) \quad \text{(B1)}$$

$$\therefore \frac{3.6 - \frac{4.5144}{\sqrt{n}}}{2.4/\sqrt{n}} = 2.054 \quad \text{(M1)}$$

$\Rightarrow \sqrt{n} = 2.62\dot{3}$ (1) $\quad n = 6.882$ (1) $\quad$ ie take $n = 7$ (E1) (next integer up) **8**

(ii) Power function $= P(\text{reject } H_0 \mid \mu)$ (M1)

$$= P\left(\bar{X} < 48.6 - \frac{4.5144}{\sqrt{7}} = 48.6 - 1.706 = 46.894 \,\middle|\, \bar{X} \sim N\left(\mu, \frac{2.4^2}{7}\right)\right)$$

↖ (M1)

$$= P\left(Z < \frac{46.894 - \mu}{\frac{2.4}{\sqrt{7}}} = 51.696 - 1.102\mu\right)$$

(1) (1) might ↑ be implicit

With $\mu = 45$, this gives $P(Z < 2.106)$ (1)

$= 0.9823$, ie > 0.98 as required. (1) **6**

Examiner’s Report

2617 Statistics 5

General Comments

There were 44 candidates from 11 centres – fewer centres than ever before, though the candidature is holding up rather better, not so many as last year but more than the year before.

There was a lot of good work. Some of it was very good indeed, though in other cases the work, while good, perhaps did not sparkle. Nevertheless, it continues to be pleasing to see that there are able candidates in the system at this level.

It needs to be said, however, that some candidates were extremely untidy in their explanations and indeed in the standard of basic handwriting, and this does not only apply in the present module.

Comments on Individual Questions

Q.1 This question started by seeking the usual derivation of the probability generating function for the $B(n, p)$ distribution together with its use to find the mean and variance, and followed this with a test for a binomial p parameter. Most candidates were able to make good attempts at the entire question. Sadly there were a few who clearly had no idea how to find the probability generating function, despite this being about the simplest of the standard examples. These candidates showed yet again that selective examination preparation covering only part of the syllabus in the hope of "question spotting" can be very dangerous. Proceeding to the binomial p test, many candidates made the error of not using the continuity correction, but otherwise there were few mistakes.

Value of test statistic is 1.54, refer to N(0,1), critical point 1.282.

Q.2 This question was based on the discrete distribution taking values −1, 0 and 1 each with probability 1/3. The mean [0] and variance [2/3] were written down without difficulty, and nearly all candidates also quickly obtained the moment generating function $\left[\frac{1}{3}\left(1+e^{\theta}+e^{-\theta}\right)\right]$.

The next part of the question required candidates to obtain the moment generating function of the standardised mean from this distribution. The answer was given, so that candidates could work with it in subsequent parts, and there were too many scripts in which it appeared by dubious processes often accompanied by heavy and cumbersome algebra. It should, however, also be said that many candidates were very careful and thorough in their derivations. The general result of the convolution theorem and the linear transformation result should have featured in a "state carefully", and while several candidates did so, there were others for whom this instruction was honoured more in the breach.

The question then required the exponential functions in the obtained moment generating function to be expanded so as to reveal an approximation for large n, which was usually done well. From this, candidates were invited to deduce the limiting distribution for large n by considering the limiting moment generating function which, of course, turned out to be that of N(0,1). Most candidates knew near-enough what to do here, though they tended to be reluctant to stress that the result depends on the *uniqueness* of the relationship between a distribution and its moment generating function, a uniqueness that holds even in this limiting process.

Q.3 The F test here was often very well done; errors were very occasionally to use an incorrect F distribution and more often to use an incorrect critical point. The test is 5% two-sided, so an upper 2½% point is needed (*not* an upper 5% point). [Value of test statistic is 4.999 (or 5.0 to one decimal place, which was readily accepted), refer to F with 10 and 12 degrees of freedom, critical point is 3.37.]

The one-sided confidence interval for a variance was also usually successfully obtained, but candidates seemed less secure here. Sometimes notation was poor (e.g. it should not be averred that the *population* variance follows a chi-squared distribution), and sometimes pieces of arithmetic appeared out of nowhere. An occasional candidate got the process the wrong way round and obtained a lower confidence bound. [Use of chi-squared with 10 degrees of freedom, lower 5% point is 3.94, hence upper confidence bound is 27.93.]

The final part sought a careful statement of the underlying distributional results and assumptions. While many candidates knew exactly what these are, there were several who did not, or at least only had a vague idea. This sadly indicates a rather shallow knowledge of "recipes" rather than a deeper understanding of where they come from. Incidentally, several candidates had in fact stated at least some of the distributional results earlier in their work, when setting up the test or confidence interval, but did not see how to bring their statements together into a coherent answer to this last part. Credit was given for the statements they had made, no matter where they occurred.

Q.4 There seems to have been a step backwards in candidates' understanding of this work, as this question was on the whole less well answered than questions in this area in previous years. There were many solutions in which the arguments were ill-constructed and poorly explained. Fairly generous marking on a benefit-of-doubt basis was used, but this should not mask the fact that considerable confusion evidently existed in many candidates' minds. As has been suggested elsewhere in this report, this probably indicates that many candidates lack a proper understanding of the methods for setting up tests and dealing with error structures, so that they cannot easily get beyond the standard comparatively simple examples of using tests. This does not bode too well for any higher-level mathematical or statistical work to which they might proceed. The published mark scheme is of course intended as a marking document which is not the same as a teaching/learning document, but nevertheless exhibits a fairly detailed line-by-line analysis of the situation. Candidates do not necessarily have to follow this exactly, of course, but a careful basic understanding of the situation is called for – and was often lacking. Inequalities were often got the wrong way round, or given as equalities; partly related to this, critical points were taken from the wrong end of the Normal distribution. Quite often something like the right answer emerged, but it did not seem that it had been reached by a secure and coherent route. However, having said all that, it should at once also be said that several candidates were careful and thorough and clearly in total command of the situation. One feature remained to trouble many candidates, which was to show that, having achieved $n = 6.882$, they should take a sample size of at least 7 (this being the answer given in the question). It was often not evident that n was in fact $\geqslant 6.882$ (particularly when the question had been largely worked with equalities rather than inequalities), and it was then rare to get any explanation that the requirements set out in the question would only be met if one went to the 'next integer up', as opposed perhaps to the nearest integer or the integer part of the obtained decimal answer.

The last part of the question asked for an expression for the power function. Much the same remarks apply here – some very careful analyses, but too many that were only, as one might say, "something like" the right approach.